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Energy bounds for the spinless Salpeter equation: harmonic oscillator

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Abstract

We study the eigenvalues $\mathcal{E}_{n\ell}$ of the Salpeter Hamiltonian $H = \beta\sqrt{m^2 + \boldsymbol{p}^2} + vr^2$, $v > 0$, $\beta > 0$, in three dimensions. By using geometrical arguments we show that, for suitable values of P, here provided, the simple semiclassical formula

$$\mathcal{E}_{n\ell} \approx \min_{r>0} \left\{ v(P_{n\ell}/r)^2 + \beta\sqrt{m^2 + r^2} \right\}$$

provides both upper and lower energy bounds for all the eigenvalues of the problem.

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1. Introduction

The Bethe–Salpeter formalism [1] is generally accepted, in principle, as the appropriate framework for the description of bound states within a (relativistic) quantum field theory. Unfortunately, almost all applications of this formalism face serious problems of both a conceptual and practical nature. In particular, it turns out to be a highly nontrivial task to extract exact information about the solutions. Consequently, one is led to consider some reasonable simplifications of the full Bethe–Salpeter equation, such as the following: the elimination of any dependence on time-like variables by the use of a static or instantaneous approximation to the interaction kernel, leading to the 'Salpeter equation' [2]; the neglect of the spin degrees of freedom of the bound-state constituents; and the restriction to positive-energy solutions. With these constraints one arrives at the 'spinless Salpeter equation', which may be regarded as a straightforward generalization of the nonrelativistic Schrödinger formalism towards relativistic kinematics: it describes the bound states of scalar particles as well as the spin-averaged spectra of the bound states of fermions.

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In this paper we study the Salpeter Hamiltonian

$$H = \beta\sqrt{m^2 + \boldsymbol{p}^2} + vr^2 \tag{1.1}$$

in which $\beta > 0$ is a parameter (allowing, for example, for more than one particle), m is the mass, and vr^2 is the harmonic-oscillator potential with coupling $v > 0$. In the momentum-space representation [3–7] the operator $\boldsymbol{p}$ becomes a c variable and thus, from the spectral point of view, the Hamiltonian $H = \beta\sqrt{m^2 + \boldsymbol{p}^2} + vr^2$ is equivalent to the Schrödinger operator $\mathcal{H}$ given by

$$\mathcal{H} = -v\Delta + V(r) \qquad V(r) = \beta\sqrt{m^2 + r^2}. \tag{1.2}$$

Since the potential $V(r)$ in (1.2) increases without bound, we know [8] that the spectrum of the operator $\mathcal{H}$ is entirely discrete, and we denote its eigenvalues (also the eigenvalues of H) by

$$\mathcal{E}_{n\ell}(v, \beta, m) \qquad n = 1, 2, 3, \ldots \quad \ell = 0, 1, 2, \ldots \tag{1.3}$$

where n 'counts' the radial states in each angular-momentum subspace, labelled by ℓ. Because $V(r)$ is at once a concave function of r^2 and a convex function of r, this allows us to derive in section 2 the approximation formula

$$\mathcal{E}_{n\ell} \approx \min_{r>0}\left\{v\left(\frac{P_{n\ell}}{r}\right)^2 + \beta\sqrt{m^2 + r^2}\right\} \tag{1.4}$$

which, for suitable values of P, provides both upper and lower bounds to the energy.

2. The energy bounds

If we think of the potential $V(r)$ as a smooth transformation $V(r) = g(\mathrm{sgn}(q)r^q)$ of a pure attractive power potential $\mathrm{sgn}(q)r^q$, then the potential $V(r)$ clearly has two such representations, each with definite convexity. That is to say, we may write the potential in the following way:

$$V(r) = \beta\sqrt{m^2 + r^2} = g^{(1)}(r^2) = g^{(2)}(r) \tag{2.1}$$

where the two transformation functions g have the properties that $g^{(1)}$ is concave ($g'' < 0$) and $g^{(2)}$ is convex ($g'' > 0$). These are precisely the conditions under which our 'envelope theory' [9–11] applies. The tangent lines to the transformation functions g are, on the one hand, shifted oscillator potentials of the form $a + br^2$, and, on the other hand, shifted linear potentials of the form $a + br$. The potential $V(r)$ itself is at once the envelope of an upper oscillator family and a lower linear family of potentials. It follows [11] that we may write

$$\min_{r>0}\left\{v\left(\frac{P_{n\ell}(1)}{r}\right)^2 + \beta\sqrt{m^2 + r^2}\right\} \leqslant \mathcal{E}_{n\ell} \leqslant \min_{r>0}\left\{v\left(\frac{P_{n\ell}(2)}{r}\right)^2 + \beta\sqrt{m^2 + r^2}\right\} \tag{2.2}$$

where the 'upper' and 'lower' P numbers are defined in terms of the corresponding exact power eigenvalues E as follows. Suppose the exact eigenvalues of the Schrödinger operator for the pure-power potential $-\Delta + \mathrm{sgn}(q)r^q$, $q \geqslant -1$, are written $E_{n\ell}(q)$, then the corresponding P numbers are defined [11] by

$$P_{n\ell}(q) = |E_{n\ell}(q)|^{(2+q)/2q}\left[\frac{2}{2+q}\right]^{1/q}\left[\frac{|q|}{2+q}\right]^{1/2} \qquad q \neq 0. \tag{2.3}$$

The limiting case $q \to 0$ corresponds *exactly* to the $\log(r)$ potential, but that is another story [12]. Thus we have for the harmonic-oscillator potential ($q = 2$)

$$E_{n\ell}(2) = 4n + 2\ell - 1 \qquad P_{n\ell}(2) = 2n + \ell - \tfrac{1}{2} \tag{2.4}$$

Table 1. Numerical values for the P numbers for the linear potential ($q = 1$) used in the Schrödinger eigenvalue formula (2.3).

n	ℓ	$P_{n\ell}(1)$
1	0	1.376 08
2	0	3.181 31
3	0	4.992 55
4	0	6.805 14
5	0	8.618 23
1	1	2.371 92
2	1	4.155 01
3	1	5.953 00
4	1	7.757 01
5	1	9.564 08
1	2	3.370 18
2	2	5.141 35
3	2	6.929 11
4	2	8.725 15
5	2	10.525 96
1	3	4.369 23
2	3	6.132 98
3	3	7.913 04
4	3	9.702 36
5	3	11.497 48
1	4	5.368 63
2	4	7.127 32
3	4	8.901 48
4	4	10.685 21
5	4	12.475 32

and for the linear potential ($q = 1$)

$$P_{n\ell}(1) = 2\left(\frac{E_{n\ell}(1)}{3}\right)^{3/2} \tag{2.5}$$

which are listed here in table 1. Thus we have established the principal claim of this paper. The minimizations in (2.2) can be carried out exactly, yielding the following quartic equation for r^2:

$$r^8 = \frac{4v^2P^4}{\beta^2}(m^2 + r^2). \tag{2.6}$$

However, given the ease of contemporary computing, the minimizations in (2.2) are perhaps preferable to the exact analytical solutions to (2.6); moreover, (2.2) involves meaningful semiclassical energy expressions which clearly exhibit the kinetic and potential energy contributions and how they depend on the parameters of the problem. It turns out that these energy bounds are quantitatively equivalent to the bounds obtained in the appendix of [7] by the use of optimized operator inequalities. The present formulation of the energy bounds, based on convexity, allows for the uniform and succinct expression of our upper and lower results by equation (2.2), and admits further natural generalizations.

As an illustration of the accuracy of (2.2) we have plotted graphs of the approximate eigenvalues given by (2.2) in figure 1. We illustrate the case $\beta = v = 1$ and plot the eigenvalue bounds obtained as functions of the mass m. Corresponding accurate numerical eigenvalues are shown as a broken curve between each pair of bound curves. These graphs confirm numerically

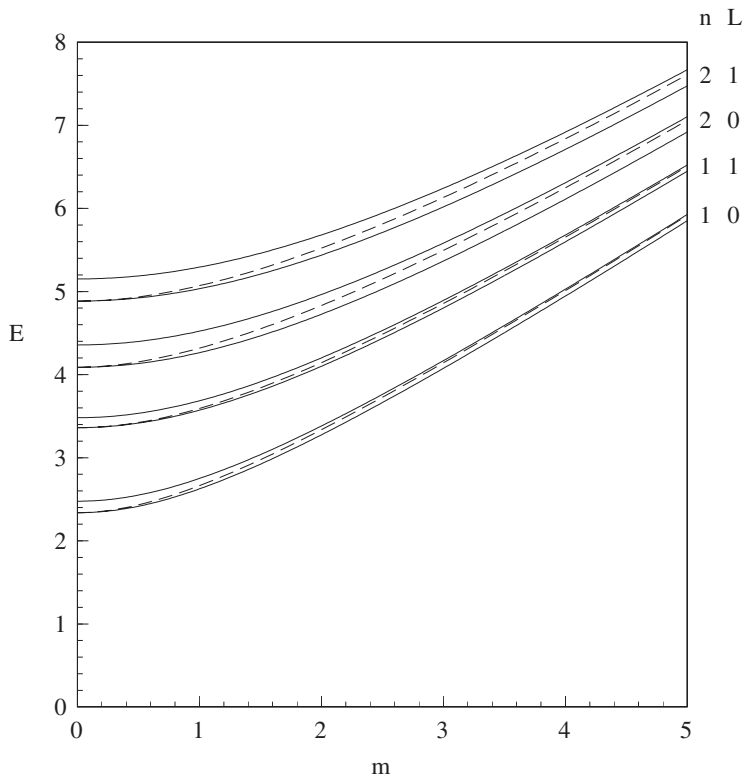

Figure 1. Upper and lower bounds on the energy levels of the spinless Salpeter Hamiltonian H with a harmonic-oscillator potential, $H = \beta\sqrt{m^2 + \boldsymbol{p}^2} + vr^2$, for $\beta = v = 1$, as functions of the mass m. Accurate numerical eigenvalues are shown as broken curves between each pair of bounds.

what is immediately clear directly from the operator H, that the corresponding (Schrödinger) problem $\mathcal{H}$ approaches the oscillator for large m and the linear potential for small m.

3. Conclusion

The spinless Salpeter eigenvalue problem is not easy to solve. Even for the harmonic-oscillator potential, one obtains an equivalent Schrödinger problem with the potential $V(r) = \beta\sqrt{m^2 + r^2}$ that does not admit exact analytical solutions in terms of known special functions. Thus, even for the oscillator problem, we must resort to approximations of some sort. The potential $V(r)$ increases monotonically to infinity and we therefore know *a priori* that the spectrum is entirely discrete. Hence, the spectrum and wave functions can be found numerically with considerable ease. In spite of this, it is always desirable to have an eigenvalue *formula*, even an approximate one, which tells us how the spectrum depends on all the parameters of the problem. In this paper we have used geometrical envelope theory to generate simple semiclassical expressions that provide upper and lower bounds to all the eigenvalues.

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References

[1] Salpeter E E and Bethe H A 1951 *Phys. Rev.* **84** 1232
[2] Salpeter E E 1952 *Phys. Rev.* **87** 328
[3] Lucha W and Schöberl F F 1996 *Phys. Rev.* A **54** 3790
[4] Lucha W and Schöberl F F 1999 *Int. J. Mod. Phys.* A **14** 2309
[5] Lucha W and Schöberl F F 1999 *Fizika* B **8** 193
[6] Lucha W and Schöberl F F 1999 *Phys. Rev.* A **60** 5091
[7] Lucha W and Schöberl F F 2000 *Int. J. Mod. Phys.* A **15** 3221
[8] Reed M and Simon B 1978 *Methods of Modern Mathematical Physics IV: Analysis of Operators* (New York: Academic)
[9] Hall R L 1983 *J. Math. Phys.* **24** 324
[10] Hall R L 1984 *J. Math. Phys.* **25** 2708
[11] Hall R L 1993 *J. Math. Phys.* **34** 2779
[12] Hall R L 2000 *J. Phys. G: Nucl. Phys.* **26** 981